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Stability of a local greedy distributed routing algorithm

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Abstract: In this work, we study the problem of routing packets between undifferentiated sources and sinks in a network modeled by a multigraph. We provide a distributed and local algorithm that transmits packets hop by hop in the network and study its behaviour. At each step, a node transmits its queued packets to its neighbours in order to optimize a local gradient. This protocol is thus greedy since it does not require to record the history about the past actions, and lazy since it only needs informations of the neighborhood.

We prove that this protocol is stable in the sense that the number of packets stored in the network stays bounded as soon as the sources injects a flow that another method could have exhausted. In particular, our protocol stays stable even if the feasibility condition is not strict on topologies with several sources and destinations, under a conjecture when the value of the flow is constrained at the destination nodes. We therefore reinforce a result from the literature that worked for differentiated suboptimal flows.

Key-words: distributed algorithm, greedy, stability

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Stabilité d'un algorithme distribué de routage de type gradient glouton local

Résumé : Dans cet article, nous nous intéressons au problème du routage depuis des sources vers des puits indifférenciés dans un réseau modélisé par un multigraphe. Nous étudions le comportement d'un algorithme distribué et local de transmission de paquets de proche en proche dans le réseau. A chaque étape, un nœud transmet les paquets qu'il a en transit vers ses voisins de manière à optimiser un gradient local. Ce protocole est ainsi glouton puisque ne prenant pas en compte l'historique du réseau, et "feignant" puisque ne considérant que l'information de ses voisins.

Nous montrons que, dans le cas de plusieurs sources et destinations indifférenciées, notre protocole est stable dans le sens où le nombre de paquets en transit dans le réseau reste borné tant que les sources injectent un flot qu'une autre méthode saurait écouler. En particulier, notre protocole reste stable même si la condition de faisabilité n'est plus stricte sur des topologies possédant plusieurs sources et destinations, sous réserve d'une conjecture dans un cas particulier où la valeur du flot est contrainte au niveau des destinations du réseau. Nous renforçons donc, sous nos hypothèses, un résultat de la littérature valable pour des flots différenciés sous-optimaux.

Mots-clés : algorithme distribué, glouton, stabilité

1 Introduction

The actual progress of networks involves an increasing interest for distributed algorithms that use only few information about the network [3]. We study a local protocol for routing packets dynamically. We prove that the protocol is *stable* , i.e. the number of packets stored at the nodes of the network is bounded (does not grow to infinity).

In previous works, Srikant et al.[6] studied distributed and localized algorithms to transmit packets in a network. In their study, they do not deal with the routing and only focus on the call scheduling for one-hop communications when calls are matching and packets enter the network continuously. They base their work on an article by Tassioulas et al. [4] who have proposed a family of stable algorithms. In both of these cases, packets are injected into the network following a stochastic process that respects a *strict* feasibility constraint, saying that the number of added packets at a time is always strictly lower than the value of the maximum flow.

Other works have considered processes in which packets are given by an adversary who wants to make the protocol fail [5]. Two distributed algorithms in dynamic networks in which topology and traffic settings can change among time have been developed [1]. In that case, the proof of stability has been done for networks with only one destination node.

In this work, we consider a simplified network model in which sources inject packets into the network, then lazy nodes forward these packets according to a local greedy gradient computation with the only information of their neighbours' state, and sinks extract the packets from the network. This behaviour can be related to the distributed algorithm for the maximum flow problem proposed by Goldberg et Tarjan [2].

We show that, in the case with undifferentiated sources and sinks, our protocol is optimal in the sense that the number of packets stored in the network stays bounded as soon as the sources injects a flow that another method could have exhausted. In particular, our protocol stays stable without the strict feasibility condition on network with several sources and destinations, under a conjecture in a specific case when the value of the flow is constrained at the destination nodes.

1.1 The network model

Let $G = (V, E)$ be a multigraph modeling the considered network. We denote Δ the maximum degree of G: $\Delta = \max_{v \in V} |\Gamma(v)|$, where $\Gamma(u)$ is the neighborhood of node $u \in V$.

To each vertex is associated a queue state which represents the number of packets waiting to be transmitted at this node. We represent this queue state by $q_t(v)$ for $v \in V$ and a given time step t, also called the *height* of v. Let $\mathcal{S} \subseteq V$ and $\mathcal{D} \subseteq V$ be respectively the sets of source and destination nodes. A such network is called $\mathcal{S}$-$\mathcal{D}$-network and is depicted on Figure 1.

The network is synchronous and at each time step:

- each source $s \in \mathcal{S}$ injects $in(s)$ packets in its queue,

- each link can transmit at most 1 packet, and this packet can be lost without any information at the sending node,
- each sink $d \in \mathcal{D}$ extracts $\min\{out(d), q_t(d)\}$ packets of its queue.

All links can eventually transmit at the same time, and the set of links that simultaneously transmit at time t is denoted E_t.

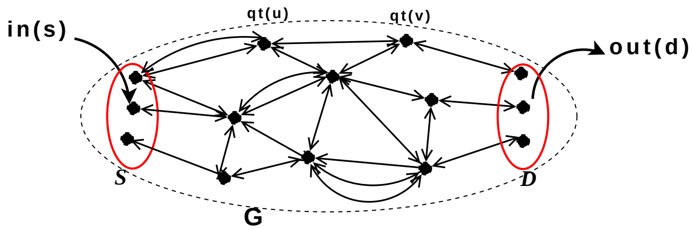

Figure 1: The multigraph G representing the network.

The arrival rate of the $\mathcal{S}$-$\mathcal{D}$-network is defined as the sum of packets injected in the source's queues at a given time step: $\sum_{s\in\mathcal{S}} in(s)$. We are interested in the total number of stored packets at a given time in the network. We quantify this number in the following definition:

Definition 1 ($\mathcal{S}$-$\mathcal{D}$-network state) *The* network state *at time t is defined by the function $P_t = \sum_{v\in V} q_t^2(v)$.*

1.2 The LGG protocol

The $\mathcal{S}$-$\mathcal{D}$-network nodes run Algorithm 1 simultaneously. They only need to get access to the queue state of their neighbours.

At each time step t, each source s injects $in(s)$ packets in its queue. Then, each node u transmits 1 packet on each of its outgoing arcs with destination v that has the smallest height, as soon as u still has packets in its queue. In particular, if u would have sent more than $q_t(u)$ packets, then it chooses to send to its $q_t(u)$ neighbours of smallest height. This choice actually has no impact on the system stability. The set of transmissions of u at time t is denoted $E_t(u)$, and $\cup_{u\in V} E_t(u) = E_t$. Packets destined to node v are deleted from u's queue, and, for each successful transmission, 1 packet is added to v's queue. Finally, each sink d remove $\min\{out(d), q_t(d)\}$ packets from its queue and step t is over.

We then introduce the notion of stability:

Definition 2 (Stability) *Given an execution of LGG in a $\mathcal{S}$-$\mathcal{D}$-network G, LGG is* stable *on G if the number of packets stored in any node of G stays bounded, i.e., the sequence $(P_t)_{t\in\mathbb{N}}$ is bounded.*

Algorithm 1: Algorithm LGG : Local greedy gradient.

$E_t(u) \leftarrow \emptyset$
$q \leftarrow q_t(u)$
$list(u) \leftarrow$ order $\Gamma(u)$ by increasing q_t
for all $v \in list(u)$ **do**
  if $q_t(u) > q_t(v)$ && $q > 0$ **then**
    $E_t(u) \leftarrow E_t(u) \cup \{(u,v)\}$
    $q \leftarrow q - 1$
$\forall (u,v) \in E_t(u)$, u sends 1 packet to v

Let G^* be the multigraph obtained from G by adding a virtual source s^* and a virtual sink d^*, with a link of infinite capacity between s^* and s for all $s \in \mathcal{S}$, and a link of capacity $out(d)$ between d and d^* for all $d \in \mathcal{D}$.

In this extended graph, we want to compute a flow Φ from s^* to d^* verifying the following constraints:

- $\Phi(e) \leqslant c(e) = \begin{cases} 1 & \forall e \in E(G) \\ out(d) & \forall (d, d^*) \\ \infty & \forall (s^*, s) \end{cases}$

- $\sum_{e \in \Gamma^+(v)} \Phi(e) = \sum_{e \in \Gamma^-(v)} \Phi(e), \ \forall v \in V(G).$

Such a flow in G^* is said *feasible* with value $f(\Phi) = \sum_{e=(s^*,s)} \Phi(e) = \sum_{e=(d,d^*)} \Phi(e)$. In the following, we denote f^* the value of a maximum s^*-d^* flow in G^*: $f^* = \max_\Phi f(\Phi)$. We can see that f^* is the value of the maximum $\mathcal{S}$-$\mathcal{D}$-flow in G given the extracting capacities $out(d)$.

We now compare the performances of our algorithm LGG with the one sending the packets on the links of a maximum flow in G^*. To do so, we define the feasibility of a $\mathcal{S}$-$\mathcal{D}$-network that states the existence of a flow with value greater than or equal the arrival rate in LGG.

Definition 3 (Feasible $\mathcal{S}$-$\mathcal{D}$-network) *A $\mathcal{S}$-$\mathcal{D}$-network G is* feasible *if it exists a s^*-d^*-flow Φ in G^* such that $in(s) \leqslant \Phi(s^*, s)$ for each source $s \in \mathcal{S}$.*

In particular, is a $\mathcal{S}$-$\mathcal{D}$-network is feasible, then it exists a maximum s^*-d^*-flow Φ of value f^* such that $in(s) \leqslant \Phi(s^*, s), \ \forall s \in \mathcal{S}$, therefore $\sum_{s \in \mathcal{S}} in(s) \leqslant f^*$. A different way to define a feasible network is to fix the capacity of links (s^*, s) to $in(s)$ and verify if a flow Φ with $in(s) = \Phi(s^*, s)$ exists in G^*. This definition is equivalent and will be preferred in Section 1.3.

The rest of the paper is about the proof of stability of LGG when the network is feasible. We first remark that the system can diverge if $\sum_{s \in \mathcal{S}} in(s) > f^*$. Indeed, without any additional assumptions on the packet loss, we can assume that all packets are delivered. It is then enough to look at a $\mathcal{S}$-$\mathcal{D}$ minimum cut (A, B) (of value f^*), with $\mathcal{S} \subseteq A$. At each step at most f^* packets leave A whereas $\sum_{s\in\mathcal{S}} in(s)(> f^*)$ enter it. So P_t strictly increases at each step. We now state the theorem of stability for a feasible $\mathcal{S}$-$\mathcal{D}$-network:

Theorem 1 *Let G be a $\mathcal{S}$-$\mathcal{D}$-network. If G is feasible, then the protocol LGG is stable on G. Otherwise, the number of packets stored in the network may diverge with the time no matter what algorithm is used.*

In the following, we can omit the adjective *feasible*, being a prerequisite for the proof of the theorem, when considering a $\mathcal{S}$-$\mathcal{D}$-network G. To prove theorem 1, we consider two cases depending on the value of the difference between the maximum flow and the arrival rate. We present separately the cases when this difference is equal to 0 and when it is strictly positive.

Definition 4 (Unsaturated $\mathcal{S}$-$\mathcal{D}$-network) *A feasible $\mathcal{S}$-$\mathcal{D}$-network G is* unsaturated *if it exists a fractional s^*-d^*-flow Φ in G^* such that $in(s) < \Phi(s^*, s)$ for each source $s \in \mathcal{S}$. Otherwise, the network is saturated.*

In other words, a $\mathcal{S}$-$\mathcal{D}$-network G is unsaturated si its arrival rate is strictly feasible, meaning that a s^*-d^*-flow is feasible in G^* when the arrival rate is set to $(1 + \epsilon)in(s)$ in each source. The unsaturated case actually corresponds to the stability region defined by Tassiulas and Ephremedis in the general case of a multicommodity flow [4].

In the next section, we prove the stability of LGG on an unsaturated $\mathcal{S}$-$\mathcal{D}$-network, covering the first part of the proof of theorem 1 when the arrival rate is strictly feasible. We then introduce in Section 2 an extended network model to deal with the proof of theorem 1 in the general case, covering the case of a saturated $\mathcal{S}$-$\mathcal{D}$-network.

1.3 Stability of an unsaturated $\mathcal{S}$-$\mathcal{D}$-network

This section is devoted to the proof of the following lemma:

Lemma 1 *If the $\mathcal{S}$-$\mathcal{D}$-network G is unsaturated, then the network state P_t is upper bounded, for all t.*

The proof is organized as follows: first, we show that the network state evolution between two consecutive steps is upper bounded. Then, we prove that, if the network state is sufficiently large at some time step, then it decreases significantly at the next step. These two properties allow to derive an upper bound for the network state for all t, leading to the stability of the protocol LGG on G.

Property 1 *The growth of the network state between two consecutive steps stays bounded for all t:*

$$P_{t+1} - P_t \leqslant 5n\Delta^2.$$

Proof. G is unsaturated, so by definition it exists a flow Φ from s^* to d^* in G^* such that, for all source $s \in \mathcal{S}$, $in(s) < \Phi(s^*, s)$.

Let us consider the evolution of the network state between step t and $t+1$:

$$\begin{aligned} P_{t+1} &= \sum_{v \in V} q_{t+1}^2(v) \\ &= \sum_{u \in V} q_t^2(v) + \sum_{v \in V} (q_{t+1}(v) - q_t(v))^2 + 2 \sum_{v \in V} q_t(v)(q_{t+1}(v) - q_t(v)). \end{aligned} \tag{1}$$

All links of G have capacity 1. So, for all $v \in V$, $(q_{t+1}(v) - q_t(v)) \leq \Delta$, where Δ is the maximum degree of G. By setting $\delta_t = \sum_{v \in V} q_t(v)(q_{t+1}(v) - q_t(v))$, we obtain:

$$P_{t+1} \leqslant P_t + 2\delta_t + n\Delta^2. \tag{2}$$

Equivalently, δ_t can be defined in function of the links in E_t used by LGG at time t for the transmissions. In the following, $e = (u,v) \in E_t$ is oriented to indicate that the packet goes from u to v. Then, δ_t can be formulated as follows:

$$\delta_t = \sum_{s \in \mathcal{S}} q_t(s) in(s) - \sum_{d \in \mathcal{D}} q_t(d) \min\{out(d), q_t(d)\} + \sum_{(u,v) \in E_t} (q_t(v) - q_t(u)). \tag{3}$$

We now compare the variation of P_t during an execution of LGG to the one obtained by pushing the packets along the paths allowing a maximum flow. Let us consider the set of paths between the sources $\mathcal{S}$ and the sinks $\mathcal{D}$ used by flow Φ, and E_t^Φ the set of links (source-to-destination oriented) of these paths selected at time t. By summing the difference of the potential on each hop along these paths, we get:

$$\sum_{(u,v) \in E_t^\Phi} (q_t(v) - q_t(u)) = -\sum_{s \in \mathcal{S}} q_t(s)\Phi(s^*, s) + \sum_{d \in \mathcal{D}} q_t(d)\Phi(d, d^*). \tag{4}$$

Let us now study the sum of the difference of the potential on the links used by LGG:

$$\begin{aligned} \sum_{(u,v) \in E_t} (q_t(v) - q_t(u)) = \sum_{(u,v) \in E_t^\Phi} (q_t(v) - q_t(u)) &- \sum_{(u,v) \in E_t^\Phi \setminus E_t} (q_t(v) - q_t(u)) \\ &+ \sum_{(u,v) \in E_t \setminus E_t^\Phi} (q_t(v) - q_t(u)). \end{aligned}$$

By definition of LGG, for all $e = (u,v) \in E_t$, $q_t(v) - q_t(u) < 0$. So, $\sum_{(u,v) \in E_t \setminus E_t^\Phi} (q_t(v) - q_t(u)) < 0$. Moreover if $e = (u,v) \in E_t^\Phi \setminus E_t$, then, again by definition of LGG, either $q_t(v) \geq q_t(u)$ or $q_t(u) \leq \Delta$. Indeed if $q_t(v) < q_t(u)$, our algorithm must send 1 packet from u to v, unless u has already sent all its available packets in $q_t(u)$. So:

$$\sum_{(u,v) \in E_t^\Phi \setminus E_t} (q_t(v) - q_t(u)) \geqslant \sum_{(u,v) \in E_t^\Phi \setminus E_t} (-\Delta) \geqslant -n\Delta^2$$

et

$$\sum_{(u,v)\in E_t}(q_t(v)-q_t(u)) \leqslant -\sum_{s\in\mathcal{S}} q_t(s)\Phi(s^*,s) + \sum_{d\in\mathcal{D}} q_t(d)\Phi(d,d^*) + n\Delta^2.$$

From equation 3, we deduce that for all t:

$$\begin{aligned}\delta_t \leqslant\ & \sum_{s\in\mathcal{S}} q_t(s)(in(s)-\Phi(s^*,s)) \\ & +\sum_{d\in\mathcal{D}} q_t(d)(\Phi(d,d^*)-\min\{out(d),q_t(d)\}) + n\Delta^2.\end{aligned} \tag{5}$$

As the network is unsaturated, by definition of Φ, $in(s) < \Phi(s^*,s)$ for all $s\in\mathcal{S}$. The sum of the queue lengths on the sources of G contributes negatively to the upper bound of δ_t. So we can neglect it. The same happens with the sum on the sinks if either $\min\{out(d),q_t(d)\} = q_t(d)$ and $\Phi(d,d^*) \leqslant q_t(d)$, or $\min\{out(d),q_t(d)\} = out(d)$. The latter case happens when $\min\{out(d),q_t(d)\} = q_t(d)$ and $\Phi(d,d^*) > q_t(d)$. Since links of G have capacity 1, Φ is bounded by Δ, leading to $\sum_{d\in\mathcal{D}} q_t(d)(\Phi(d,d^*)-\min\{out(d),q_t(d)\}) \leqslant n\Delta^2$.

We finally obtain an upper bound for δ_t: $\delta_t \leq 2n\Delta^2$. In particular, from Inequality 2, we upper bound the difference of the network state between step $t+1$ and t:

$$P_{t+1}-P_t \leqslant 5n\Delta^2.$$

□

Let introduce the value $\epsilon = \min_{s\in\mathcal{S}}(\Phi(s^*,s)-in(s))$ that is strictly positive by definition of an unsaturated network.

Property 2 *Let $Y = (\frac{5nf^*}{\epsilon}+3n)\Delta^2$. If P_t is sufficiently large, i.e. $P_t > nY^2$, then at the next step, the number of stored packets in the network strictly decreases:*

$$P_{t+1}-P_t < -5n\Delta^2.$$

Proof. From Inequality 2, the proof is equivalent to show that, if $P_t > nY^2$, then $\delta_t < -3n\Delta^2$. The rest of the proof is divided into two parts depending on the existence of a node with large height in the network.

Let us first assume that it exists a source $s\in\mathcal{S}$ such that $q_t(s) \geq \frac{5n}{\epsilon}\Delta^2$. Then, using Inequality 5 and the unsaturated property of the network, we can upper bound δ_t and prove the first part of Property 2:

$$\delta_t \leqslant -\epsilon q_t(s) + 2n\Delta^2 < -3n\Delta^2.$$

Secondly, we are now in the case where $q_t(s) < \frac{5n}{\epsilon}\Delta^2$ for all $s\in\mathcal{S}$. If $P_t \geqslant nY^2$, then it exists $x\in V\setminus\mathcal{S}$ such that $q_t(x)\geq Y$. Let $x = u_1,u_2,\cdots,u_k$ be a path from x to u_k such that $u_k = d\in\mathcal{D}$ (maybe $d=x$). Then:

$$\sum_{i<k,\ q_t(u_i)>q_t(u_{i+1})} (q_t(u_{i+1})-q_t(u_i)) - q_t(u_k)\min\{out(u_k),q_t(u_k)\} \leqslant -q_t(x).$$

This sum contributes negatively to $\sum_{(u,v)\in E_t}(q_t(v)-q_t(u)) - \sum_{d\in\mathcal{D}} q_t(d)\min\{out(d), q_t(d)\}$ (recall that the terms of the first part of the sum are negative since they are used by LGG).

From equation 3, we thus obtain that:

$$\begin{aligned}\delta_t &\leqslant \sum_{s\in\mathcal{S}} q_t(s)in(s) - q_t(x)\\ &< f^* \cdot \max_{s\in\mathcal{S}} q_t(s) - q_t(x) \leqslant f^* \cdot \frac{5n}{\epsilon}\Delta^2 - q_t(x) \leqslant -3n\Delta^2.\end{aligned}$$

By injecting the bound of δ_t in Inequality 2, we obtain a strict upper bound on the network state evolution:

$$P_{t+1} - P_t < -5n\Delta^2$$

which finishes the proof of Property 2. □

From Properties 1 and 2 we deduce that, for all t, $P_t \leqslant nY^2 + 5n\Delta^2$ which bounds the number of packets stored in the network at each time step and prove the strict stability of our algorithm. We remark that the packet losses here only improve the protocol stability.

In the case of a saturated $\mathcal{S}$-$\mathcal{D}$-network in which a flow with value $(1+\epsilon)in(s)$ on each link (s^*, s) in unfeasible, then the previous techniques do not permit to control the variations of the second derivative in Equation 1. In order to tackle these phenomena, we must generalize the network behaviour before addressing the proof of stability by induction on the network size. This generalization is presented in the next section, through the definition of the R-generalized $\mathcal{S}$-$\mathcal{D}$-networks.

2 R-generalized $\mathcal{S}$-$\mathcal{D}$-networks

Recall that $G = (V, E)$ is the feasible $\mathcal{S}$-$\mathcal{D}$-network considered in previous sections, and (A, B) a minimum cut in G^*: (A, B) is a node partition in G^* such that $s^* \in A$, $d^* \in B$, and the sum of the link capacities between A and B is minimum. In the ideal case in which $\mathcal{S} \subseteq A$ and $\mathcal{D} \subseteq B$, (A, B) is called a $\mathcal{S}$-$\mathcal{D}$-cut, as depicted in Figure 2.

The general proof of Theorem 1 is done by induction on the network size $|V|$. To do so, we need to define a generalized network in order to model the special behaviour of the nodes located in the border of the cut (A, B) and apply the induction hypothesis. We then define the R-generalized $\mathcal{S}$-$\mathcal{D}$-networks, in which $R \geq 0$ is a constant, and such that every classical $\mathcal{S}$-$\mathcal{D}$-network is a 0-generalized $\mathcal{S}$-$\mathcal{D}$-network in the new model. The purpose of the induction is to prove that, for all $R \geq 0$ and in any *feasible* R-generalized $\mathcal{S}$-$\mathcal{D}$-network G, our protocol is stable. In particular, this prooves that LGG is stable in any $\mathcal{S}$-$\mathcal{D}$-network.

More precisely, we successively prove that parts A and B of the cut acts as generalized $\mathcal{S}$-$\mathcal{D}$-networks for well chosen constants, allowing us to apply the induction hypothesis. Several cases must be considered, depending on the location of the links between A and B that can be in G, or incident to the virtual nodes s^* and d^* added in G^*. This latter case correspond to our induction basis and is tackle in Sections 3.1 et 3.2.

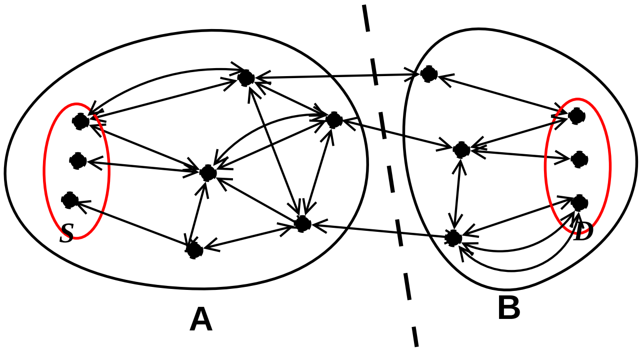

Figure 2: A minimum $\mathcal{S}$-$\mathcal{D}$-cut in the $\mathcal{S}$-$\mathcal{D}$-network G.

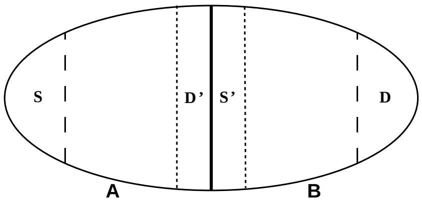

Figure 3: A $\mathcal{S}$-$\mathcal{D}$ cut and associated subsets $\mathcal{S}'$ and $\mathcal{D}'$.

The generalization of the network behaviour is needed when the cut (A, B) is located in G. In a first time, we can then remark that partition B can be viewed as a particular case of a $\mathcal{S}'$-$\mathcal{D}$-network, in which $\mathcal{S}'$ is the set of nodes in B adjacent to a node in A (Fig. 3). Each of these nodes $s' \in \mathcal{S}'$ corresponds to a source node in B that injects at most $|\Gamma_{|A}(s')| + in(s')$ packets in its queue at each step, where $\Gamma_{|A}(s')$ represents the neighborhood of s' in A, and $in(s') > 0$ in the case of $s' \in \mathcal{S}$ in G. The hypothesis of the random packet losses validates the case when s' sends packets to a node that is located in partition A. Similarly, if $s' \in \mathcal{D}$ is a destination that extracts some packets out of the network, then the extracted packets can be viewed as lost in the original network. In order to generalized the behaviour, we define pseudo-sources whose behaviour is less constrained than the one of the classical sources previously defined.

Definition 5 (Pseudo-source) *A* pseudo-source *s injects* at most *$in(s) > 0$ packets in its queue at the beginning of each step.*

This definition will be then used to prove that the number of stored packets in partition B stays bounded.

In a second time, we suppose that the number of stored packets in B is bounded by some constant R, and show that partition A can also be viewed as a $\mathcal{S}$-$\mathcal{D}'$-network, in which $\mathcal{D}'$ contains all nodes in A that has at least one neighbour in B (Fig. 3). Each $d' \in \mathcal{D}'$ has the following behaviour: if the queue length of d' is high enough ($q_t(d') > R$ for some constant R), then d' extracts at least $\min\{|\Gamma_{|B}(d')| + out(d'), q_t(d') - R\}$ packets out of its queue (since d' is higher than all its neighbours in B). Moreover, since the nodes in $\mathcal{D}'$ whose height is lower than R may receive some packets from nodes in B, their behaviour towards A can be viewed as if they could hide some packets of their queue to the nodes in A. In other words, for each $d' \in \mathcal{D}'$ such that $q_t(d') \leqslant R$, d' may declare a height $q'_t(d') \leqslant R$ to nodes in A, generalizing the behaviour of the destination nodes.

Definition 6 (R-pseudo-destination) *A* generalized destination d *extracts* at most $out(()d) > 0$ *packets of its queue at the end of each step, and, given a constant of retention* $R \geq 0$*:*

(i) if $q_t(d) > R$*, then* d *extracts at least* $\min\{out(d), q_t(d) - R\}$ *packets of its queue,*

(ii) for each $u \in \Gamma(d)$*,* d *reveals a queue size* $q'_t(d)$ *defined as follows:*

- *if* $q_t(d) > R$*, then* d *declare* $q'_t(d) = q_t(d)$*,*
- *if* $q_t(d) \leqslant R$*, then* d *declare an height* $q'_t(d) \leqslant R$*.*

We now combine these two definitions useful for the proof by induction, leading to a generalization of the network model into a R-generalized $\mathcal{S}$-$\mathcal{D}$-network that contains a set of R-generalized sources and destinations defined as follows:

Definition 7 (R-generalized source/destination) *Let* $R \geqslant 0$*, a* R*-generalized node* v *injects at most* $in(v) > 0$ *packets in its queue at the beginning of each step, extracts at most* $out(v) > 0$ *packets of its queues at the end of each step, and:*

(i) if $q_t(d) > R$*, then* d *extracts at least* $\min\{out(d), q_t(d) - R\}$ *packets of its queue,*

(ii) for each $u \in \Gamma(d)$*,* d *reveals a queue size* $q'_t(d)$ *defined as follows:*

- *if* $q_t(d) > R$*, then* d *declare* $q'_t(d) = q_t(d)$*,*
- *if* $q_t(d) \leqslant R$*, then* d *declare an height* $q'_t(d) \leqslant R$*.*

If $in(v) \leqslant out(v)$*, then* v *is called a* R-generalized destination*, otherwise it is a* R-generalized source.

Definition 8 (R-generalized $\mathcal{S}$-$\mathcal{D}$-network) *A* R*-generalized* $\mathcal{S}$*-*$\mathcal{D}$*-network is a multigraph* G *containing a set* $\mathcal{S}$ *of* R*-generalized sources, and a set* $\mathcal{D}$ *of* R*-generalized destinations. All the other nodes of* G *(*$v \in V \setminus (\mathcal{S} \cup \mathcal{D})$*) keep their "classical" behaviour, i.e. the same as in the* $\mathcal{S}$*-*$\mathcal{D}$*-network defined in Section 1.1.*

Remark 1 *Every node v of the network that is not in $\mathcal{S} \cup \mathcal{D}$ is set with $in(v) = out(v) = 0$. Nevertheless, these values may change during the induction process, and then v may become a R-generalized source or destination.*

A $\mathcal{S}$-$\mathcal{D}$-network is clearly a 0-generalized $\mathcal{S}$-$\mathcal{D}$-network. Indeed, from definition 7, 0-generalized sources and destinations have the following properties:

- a source s injects at most $in(s)$ packets in its queue at the beginning of each step,
- a destination d extracts at most $out(d)$ packets, and at least $\min\{out(d), q_t(d)\}$ packets of its queue at the end of each step (since $R = 0$ brings us in the (i)'s case of definition 7). It never lies on its queue length since $q_t(d)$ is always greater than or equal to 0, so greater to R.

Packet losses model the ability of source s to inject less than $in(s)$ in the network. In return, the behaviour is the same as in a $\mathcal{S}$-$\mathcal{D}$-network.

A R-generalized $\mathcal{S}$-$\mathcal{D}$-network G is *feasible* if its arrival rate is, i.e. if it exists a s^*-d^*-flow Φ such that $in(v) \leqslant \Phi(s^*, v)$ for all $v \in \mathcal{S} \cup \mathcal{D}$, where s^* and d^* virtual nodes added to G, defining an extended generalized network G^* (similar than in Section 1.1). In particular, G is feasible if it exists a feasible flow Φ in G^* in which links (s^*, v) have capacity $in(v)$, $\forall v \in \mathcal{S} \cup \mathcal{D}$: then, $\Phi(s^*, v) = in(v)$, $\forall v \in \mathcal{S} \cup \mathcal{D}$ (Figure 4). A R-generalized $\mathcal{S}$-$\mathcal{D}$-network G is *unsaturated* if it exists a feasible s^*-d^*-flow Φ in G^* in which links (s^*, v) have capacity $(1 + \epsilon)in(v)$, $\forall v \in \mathcal{S} \cup \mathcal{D}$. Equivalently, we can come down to definition 4 by setting an infinite capacity to links (s^*, v) in G^*.

3 Stability of a R-generalized $\mathcal{S}$-$\mathcal{D}$-network

This section is devoted to the proof of stability of our algorithm LGG on a feasible R-generalized $\mathcal{S}$-$\mathcal{D}$-network, under the following conjecture:

Conjecture 1 *If our protocol is stable in a feasible R-generalized $\mathcal{S}$-$\mathcal{D}$-network in which the generalized sources $s \in \mathcal{S}$ injects exactly $in(s)$ packets in their queue at each step, and when no packet loss is allowed, then LGG is stable if the sources can inject less than $in(s)$ packets at each time, and in the presence of packet losses.*

It is indeed common to think that the packet generation process follows a domination scheme: removing some packets does not lead to a system divergence. If the sequence $in_t(v)$ is strictly greater than another sequence $in'_t(v)$ at each time t and for all node $v \in \mathcal{S} \cup \mathcal{D}$, then the system will tend to diverge more with in_t than with in'_t.

Subject to the correctness of this conjecture, we then prove the following theorem:

Theorem 2 *For all constante $R \geqslant 0$, and in any feasible R-generalized $\mathcal{S}$-$\mathcal{D}$-network G, the protocol LGG is stable. Moreover, LGG is stable in any feasible $\mathcal{S}$-$\mathcal{D}$-network.*

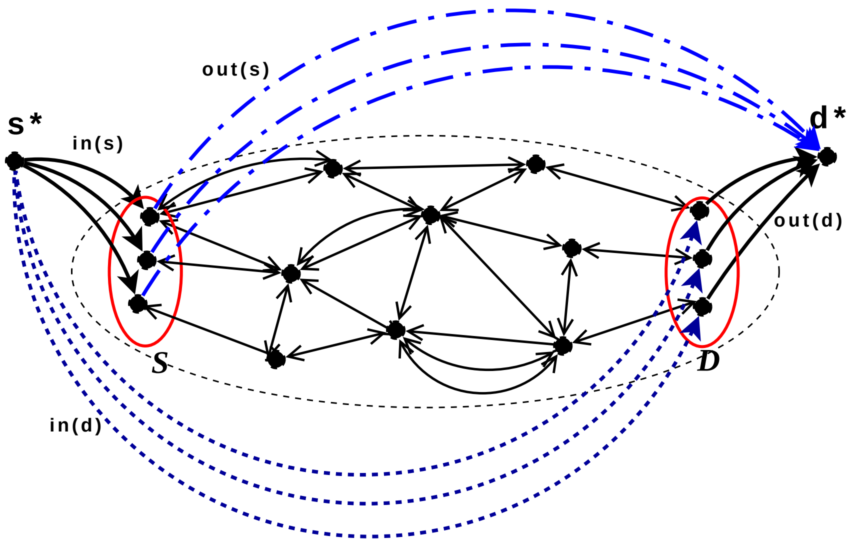

Figure 4: Un $\mathcal{S}$-$\mathcal{D}$-réseau R-généralisé étendu G^*.

This theorem is a rephrasing of theorem 1 in order to prove the stability of LGG in the saturated R-generalized $\mathcal{S}$-$\mathcal{D}$-networks, for all $R \geqslant 0$. As we showed in previous section that a $\mathcal{S}$-$\mathcal{D}$-network is a 0-generalized $\mathcal{S}$-$\mathcal{D}$-network, the proof of stability of LGG for the saturated $\mathcal{S}$-$\mathcal{D}$-networks will be complete.

Let us consider a feasible R-generalized $\mathcal{S}$-$\mathcal{D}$-network $G = (V, E)$, with $R \geqslant 0$. If $|V| = 1$, our protocol is obviously stable. Suppose now that $|V| > 1$. We define Φ as a maximum s^*-d^*-flow in G^* such that $in(v) = \Phi(s^*, v)$, $\forall v \in \mathcal{S} \cup \mathcal{D}$, and (A, B) a minimum cut in G^* of value $|(A, B)| = \sum_{v \in \mathcal{S} \cup \mathcal{D}} in(v)$. We then have to deal with three cases:

1. such a cut (A, B) is unique and corresponds to $(\{s^*\}, (V \cup \{d^*\}) \setminus \{s^*\})$: we prove in Section 3.1 that G is unsaturated, and that our protocol is stable by an adaptation of the proof of Section 1.3,

2. one other cut exists and corresponds to $((V \cup \{s^*\}) \setminus \{d^*\}, \{d^*\})$: we prove in Section 3.2 the stability of LGG subject to the correctness of Conjecture 1,

3. it exists such a cut (A, B) in G: we prove in Section 3.3 the stability by induction on the size of G, as introduced at the beginning of the previous section.

Cases 1 and 2 correspond to the basis of our induction.

3.1 Unsaturated R-generalized $\mathcal{S}$-$\mathcal{D}$-network

The first case when the unique minimum cut (A, B) in G^* is $A = \{s^*\}$, this means that the flow is only constrained by the amount of packets injected by the virtual source s^*, i.e. by the capacities $in(v)$ on links (s^*, v), $\forall v \in \mathcal{S} \cup \mathcal{D}$. Then, it exists a constant $\epsilon > 0$ allowing a feasible flow Φ in G^* in which the arrival rate is $(1+\epsilon)in(v)$ in each nœud $v \in \mathcal{S} \cup \mathcal{D}$. This is the definition of an unsaturated network as defined for a classical $\mathcal{S}$-$\mathcal{D}$-network in Section 1.1.

So we slightly change the proof of Section 1.3 to adapt to the generalized sources and sinks. In the following, we focus on the main differences of the proof. Recall that the flow Φ that does not saturate the network is a feasible s^*-d^*-flow such that $in(v) < \Phi(s^*, v)$ for all $v \in \mathcal{S} \cup \mathcal{D}$. The lemma we prove in this case is the following:

Lemma 2 *If the R-generalized $\mathcal{S}$-$\mathcal{D}$-network G is unsaturated, then the network state P_t is upper bounded for all t.*

As in Section 1.3, we decompose the proof of the lemma into two properties bounding the difference of the network state between two consecutive steps. The first property upper bounds the growth of the network state.

Property 3 *L'accroissement de l'état du réseau entre deux étapes successives reste borné pour tout t:*

$$P_{t+1} - P_t \leqslant 2|\mathcal{S} \cup \mathcal{D}|(R + out_{max})out_{max} + \Delta^2(3n - 2|\mathcal{S} \cup \mathcal{D}|) + 4|\mathcal{S} \cup \mathcal{D}|\Delta R,$$

où $out_{max} = \max_{v \in \mathcal{S} \cup \mathcal{D}} out(v)$.

Proof. From equation 1, we seek to upper bound δ_t. At each time step t, a R-generalized source/destination v in G injects $in_t(v) \leqslant in(v) < \Phi(s^*, v)$ packets in its queue. Similarly, v extracts $out_t(v) \leqslant \min\{out(v), q_t(v)\}$ packets of its queue, with $out_t(v) \geqslant \min\{out(v), q_t(v)\}$ if $q_t(v) > R \geqslant \Delta$, and $out_t(v) \geqslant 0$ otherwise.

In this condition, the first change occurs in equation 3 that becomes:

$$\begin{aligned} \delta_t = \sum_{(u,v) \in E_t} (q_t(v) - q_t(u)) \quad &+ \sum_{s \in \mathcal{S}} q_t(s)(in_t(s) - out_t(s)) \\ &- \sum_{d \in \mathcal{D}} q_t(d)(out_t(d) - in_t(d)). \end{aligned} \tag{6}$$

We now compare the value of the sum of the difference of the queue sizes on the links used by LGG and those following a maximum flow. As in Section 1.3, we decompose the sum depending on the belonging of the links to E_t^Φ, $E_t^\Phi \setminus E_t$, or $E_t \setminus E_t^\Phi$. The generalization of the netwokr does not change the behaviour of the flow. Nevertheless, as R-generalized sources and destinations both inject and extract packets, we get a different equation 4 :

$$\begin{aligned}
&\sum_{(u,v)\in E_t^\Phi} (q_t(v)-q_t(u)) \\
&\quad = \sum_{d\in\mathcal{D}} q_t(d)(\Phi(d,d^*)-\Phi(s^*,d)) - \sum_{s\in\mathcal{S}} q_t(s)(\Phi(s^*,s)-\Phi(s,d^*)) \\
&\quad = \sum_{v\in\mathcal{S}\cup\mathcal{D}} q_t(v)(\Phi(v,d^*)-\Phi(s^*,v))
\end{aligned}$$

By definition of LGG, $q_t(v)-q_t(u)<0$ for $e=(u,v)\in E_t\setminus E_t^\Phi$, except if $u\in\mathcal{S}\cup\mathcal{D}$ ou $v\in\mathcal{S}\cup\mathcal{D}$ lies on its queue size. Suppose that v is lying, then $q_t(u)\leqslant q_t(v)\leqslant R$. But v reveals a queue size $q'_t(v)\leqslant q_t(u)$, so the difference $q_t(v)-q_t(u)\leqslant R$. Suppose now that v is not lying, then u is lying and declares a height $R\geqslant q'_t(u)\geqslant q_t(v)$, which also bounds the difference $q_t(v)-q_t(u)$ by R.

So for each neighbour u of $v\in\mathcal{S}\cup\mathcal{D}$, the difference $q_t(v)-q_t(u)$ is upper bounded by R, and so is the difference for each v neighbour of $u\in\mathcal{S}\cup\mathcal{D}$, leading to the following inequality:

$$\sum_{(u,v)\in E_t\setminus E_t^\Phi} (q_t(v)-q_t(u)) \leqslant 2|\mathcal{S}\cup\mathcal{D}|\Delta R.$$

Considering the links of the flow that are not used by LGG: $e=(u,v)\in E_t^\Phi\setminus E_t$, only the case where $q_t(v)-q_t(u)\leqslant 0$ matters (it contributes positively to $\sum_{(u,v)\in E_t}(q_t(v)-q_t(u))$ that we want to bound). If a link (u,v) such that $q_t(v)\leqslant q_t(u)$ is not used by LGG, it means that:

- $q_t(u)\leqslant\Delta$ and u has already sent all its packets to its neighbours of smaller height. Then, the difference $q_t(v)-q_t(u)$ is lower bounded by $-\Delta$ as we saw in Section 1.3.
- $u\in\mathcal{S}\cup\mathcal{D}$ lies on its queue size. Therefore $q_t(v)\leqslant q_t(u)\leqslant R$, but u declare a height $q'_t(u)<q_t(v)$. Then $q_t(v)-q_t(u)\geqslant q_t(v)-R\geqslant -R$.
- $u\in\mathcal{S}\cup\mathcal{D}$ does not lie. So v lies and declare $R\geqslant q'_t(v)>q_t(u)$. The difference $q_t(v)-q_t(u)$ is always lower bounded by $-R$.

Thus, the difference of the queue sizes $q_t(v)-q_t(u)$ is lower bounded by $-R$ if $u\in\mathcal{S}\cup\mathcal{D}$ or $v\in\mathcal{S}\cup\mathcal{D}$, and by $-\Delta$ otherwise, leading to the following lower bound:

$$\sum_{(u,v)\in E_t^\Phi\setminus E_t} (q_t(v)-q_t(u)) \geqslant -\Delta^2(n-2|\mathcal{S}\cup\mathcal{D}|)-2|\mathcal{S}\cup\mathcal{D}|\Delta R.$$

We then integrate these bounds into the difference of the queue sizes on links of LGG and obtain:

$$\begin{aligned}
\sum_{(u,v)\in E_t} (q_t(v)-q_t(u)) \quad \leqslant \quad & \sum_{v\in\mathcal{S}\cup\mathcal{D}} q_t(v)(\Phi(v,d^*)-\Phi(s^*,v)) \\
& +\Delta^2(n-2|\mathcal{S}\cup\mathcal{D}|)+4|\mathcal{S}\cup\mathcal{D}|\Delta R.
\end{aligned}$$

Then, for δ_t (Eq. 6), we finally get:

$$\begin{aligned}\delta_t \leqslant & \sum_{v\in\mathcal{S}\cup\mathcal{D}} q_t(v)\left((in_t(v)-\Phi(s^*,v))-(out_t(v)-\Phi(v,d^*))\right) \\ & +\Delta^2(n-2|\mathcal{S}\cup\mathcal{D}|)+4|\mathcal{S}\cup\mathcal{D}|\Delta R.\end{aligned} \tag{7}$$

From this equation, we deduce some properties:

- $\forall s \in \mathcal{S}$, $\Phi(s^*, s) > in(s)$ since the R-generalized $\mathcal{S}$-$\mathcal{D}$-network is unsaturated, and $in(s) \geqslant in_t(s)$ from the definition of a R-generalized source. Then, $in_t(v)-\Phi(s^*,v) < 0$ for all $v \in \mathcal{S}$.
- $\forall d \in \mathcal{D}$, $\Phi(s^*, d) \geqslant in(d)$ from the definition of a feasible network.
- $\forall v \in \mathcal{S}\cup\mathcal{D}$: if $\Phi(v, d^*) \leqslant out_t(d)$, then

  $$\delta_t \leqslant \Delta^2(n-2|\mathcal{S}\cup\mathcal{D}|)+4|\mathcal{S}\cup\mathcal{D}|\Delta R.$$

  On the contrary, if $\Phi(v,d^*) > out_t(d)$, then $q_t(v) \leqslant R + out(v)$. Indeed, if $q_t(v) > R + out(v)$, then $out_t(v) \geqslant \min\{q_t(v)-R, out(v)\} = out(v) \geqslant \Phi(v,d^*)$, which leads to a contradiction. Thus,

  $$\begin{aligned}\delta_t &\leqslant \sum_{v\in\mathcal{S}\cup\mathcal{D}} (R+out(v))out(v) + \Delta^2(n-2|\mathcal{S}\cup\mathcal{D}|)+4|\mathcal{S}\cup\mathcal{D}|\Delta R \\ &\leqslant |\mathcal{S}\cup\mathcal{D}|(R+\max_{v\in(\mathcal{S}\cup\mathcal{D})} out(v)) \max_{v\in(\mathcal{S}\cup\mathcal{D})} out(v) \\ &\quad +\Delta^2(n-2|\mathcal{S}\cup\mathcal{D}|)+4|\mathcal{S}\cup\mathcal{D}|\Delta R\end{aligned}$$

We have derived an upper bound for δ_t that is independent of t. From inequality 2, the growth of the network state between two consecutive steps is therefore upper bounded. □

The second property needed to prove the stability of LGG on the unsaturated R-generalized $\mathcal{S}$-$\mathcal{D}$-network is the following:

Property 4 *Given a constant Y large enough, if $P_t > nY^2$, then at the newt step, the number of stored packets in the network strictly decreases:*

$$P_{t+1} - P_t < -2|\mathcal{S}\cup\mathcal{D}|(R+out_{max})out_{max} - \Delta^2(3n-2|\mathcal{S}\cup\mathcal{D}|) - 4|\mathcal{S}\cup\mathcal{D}|\Delta R,$$

where $out_{max} = \max_{v\in\mathcal{S}\cup\mathcal{D}} out(v)$.

Proof. Let $A = 2|\mathcal{S}\cup\mathcal{D}|(R+out_{max})out_{max} + \Delta^2(3n-2|\mathcal{S}\cup\mathcal{D}|)+4|\mathcal{S}\cup\mathcal{D}|\Delta R$. From inequality 2, proving Property 4 is equivalent to show that, if $P_t \geqslant nY^2$, then $\delta_t < -\frac{A+n\Delta^2}{2}$. As in Section 1.3, the proof follows two cases.

First, suppose that it exists a generalized node $x \in \mathcal{S}\cup\mathcal{D}$ such that

$$q_t(x) > \frac{\Delta^2(3n-2|\mathcal{S}\cup\mathcal{D}|)+7|\mathcal{S}\cup\mathcal{D}|R\Delta+|\mathcal{S}\cup\mathcal{D}|(R+out_{max})out_{max}}{\epsilon}.$$

Recall that $\epsilon = \min_{v\in\mathcal{S}\cup\mathcal{D}}(\Phi(s^*,v) - in(v))$, then from equation 7 and the fact that $in_t(v) - \Phi(s^*,v) \leqslant in(v) - \Phi(s^*,v) < -\epsilon$ for all $v \in \mathcal{S}\cup\mathcal{D}$ and for $\epsilon > 0$, we get as in Section 1.3:

$$\begin{array}{rl}\delta_t & \leqslant -\epsilon q_t(s) + \Delta^2(n - |\mathcal{S}\cup\mathcal{D}|) + 5|\mathcal{S}\cup\mathcal{D}|\Delta R \\ & < -\frac{A+n\Delta^2}{2}\end{array}$$

which proves the first part of Property 4.

Second, we have $q_t(v) \leqslant \frac{\Delta^2(3n-2|\mathcal{S}\cup\mathcal{D}|)+7|\mathcal{S}\cup\mathcal{D}|R\Delta+|\mathcal{S}\cup\mathcal{D}|(R+out_{max})out_{max}}{\epsilon}$, $\forall v \in \mathcal{S}\cup\mathcal{D}$. In this case, if $P_t \geqslant nY^2$, then it exists a node $x \in V \setminus (\mathcal{S}\cup\mathcal{D})$ with large height and a path defined as in Section 1.3. The sum of the difference of the queue sizes along this path is thus: $\sum_{i<k,\ q_t(u_i)>q_t(u_{i+1})}(q_t(u_{i+1}) - q_t(u_i)) \leqslant q_t(u_k) - q_t(x)$.

Recall that for all $e = (u,v) \in E_t$ with either $v \notin \mathcal{S}\cup\mathcal{D}$, or $v \in \mathcal{S}\cup\mathcal{D}$ and $q_t(v) > R$ (respectively with either $u \notin \mathcal{S}\cup\mathcal{D}$, or $u \in \mathcal{S}\cup\mathcal{D}$ and $q_t(u) > R$), we have $q_t(v) - q_t(u) < 0$. Moreover, for all $e = (u,v) \in E_t$ such that $v \in \mathcal{S}\cup\mathcal{D}$ and $q_t(v) \leqslant R$ (respectively $u \in \mathcal{S}\cup\mathcal{D}$ and $q_t(u) \leqslant R$), we have $q_t(v) - q_t(u) \leqslant R$.

The sum along links used by LGG is thus bounded:

$$\sum_{(u,v)\in E_t}(q_t(v) - q_t(u)) \leqslant 2|\mathcal{S}\cup\mathcal{D}|\Delta R + q_t(u_k) - q_t(x).$$

We also get the following lower bound:

$$\sum_{d\in\mathcal{D}} q_t(d)(out_t(d) - in_t(d)) \geqslant \sum_{d\in\mathcal{D}} q_t(d) - (R+1)f^*$$

because if $q_t(d) \leqslant R$ for a R-generalized destination d, then $out_t(d) \leqslant in_t(d)+1$ by definition. Moreover, by definition of the value of the maximum s^*-d^*-flow f^*, $\sum_{d\in\mathcal{D}} in_t(d) \leqslant f^*$.

From equation 6, we therefore get:

$$\begin{array}{rl}\delta_t & \leqslant \displaystyle\sum_{s\in\mathcal{S}} q_t(s)(in_t(s) - out_t(s)) + (R+1)f^* \\ & \quad -\displaystyle\sum_{d\in\mathcal{D}} q_t(d)(out_t(d) - in_t(d)) + 2(|\mathcal{S}\cup\mathcal{D}|)\Delta R + q_t(u_k) - q_t(x) \\ & \leqslant \displaystyle\sum_{s\in\mathcal{S}} q_t(s)(in_t(s) - out_t(s)) + (R+1)f^* + 2|\mathcal{S}\cup\mathcal{D}|\Delta R - q_t(x).\end{array}$$

By choosing Y sufficiently large such that $q_t(x) > Y$, Property 4 is satisfied. □

From Properties 3 and 4, we conclude that, for all t, the network state P_t is upper bounded, which limits the number of stored packets in the network at any time step and validate the stabilitiy of LGG in an unsaturated R-generalized $\mathcal{S}$-$\mathcal{D}$-network.

3.2 R-generalized $\mathcal{S}$-$\mathcal{D}$-network saturated at the destinations

We suppose here that $in_t(v) = in(v)$, $\forall v \in \mathcal{S}\cup\mathcal{D}$ and t, and there is no packet loss. We prove the stability of LGG in this particular case. Conjecture 1 allows us to conclude that

LGG is stable in the more general case in which $in_t(v) \leqslant in(v)$ and with possible packet losses.

$(\{s^*\}, (V \cup \{d^*\}) \setminus \{s^*\})$ is not the only minimum cut in G^*. A second cut exists and is located at the virtual destination d^* : $(A, B) = ((V \cup \{s^*\}), \{d^*\})$. In other words, partition B is the only node d^*. The value of the cut is $|(A, B)| = \sum_{v \in \mathcal{S} \cup \mathcal{D}} in(v) = \sum_{v \in \mathcal{S} \cup \mathcal{D}} out(v)$.

In the R-generalized $\mathcal{S}$-$\mathcal{D}$-network G, we assume that it exists a time step t_0 and a constant $R' \geqslant R + \max_{v \in \mathcal{S} \cup \mathcal{D}} out(v)$ such that for all $t \geqslant t_0$ and $v \in \mathcal{S} \cup \mathcal{D}$, $q_t(v) \geqslant R'$. From the definition of the R-generalized nodes, if $q_t(v) > R + out(v)$, then v extracts exactly $out(v)$. A each step $t \geqslant t_0$, the arrival rate in LGG is lower than or equal to the extracting rate of the R-generalized sources/destinations. The growth of the number of stored packets is thus:

$$\begin{aligned} \sum_{v \in V} q_{t+1}(v) &= \sum_{v \in V} q_t(v) - \sum_{v \in V} out(v) + \sum_{v \in V} in(v) \\ &\leqslant \sum_{v \in V} q_t(v) \leqslant \sum_{v \in V} q_{t_0}(v) \end{aligned}$$

The network state is therefore bounded along time.

On the contrary, we suppose that it exists at least a node $v \in \mathcal{S} \cup \mathcal{D}$ infinitely bounded according to the following definition:

Definition 9 (Infinitely bounded node) *A node is* infiniment borné *if it exists a constant such that its queue size go above this constant an infinite number of times. More formally, a node $v \in V$ is infinitely bounded if $\exists M > 0$ such that $\forall t_0$, $\exists t \geqslant t_0$ such that $q_t(v) \leqslant M$.*

We say that a set of nodes is infinitely bounded if all the nodes in it are infinitely bounded.

In the R-generalized $\mathcal{S}$-$\mathcal{D}$-network G, we suppose that it exists a constant $R' \geqslant R + \max_{v \in \mathcal{S} \cup \mathcal{D}} out(v)$ and a node $v \in \mathcal{S} \cup \mathcal{D}$ such that $\forall t_0$, $\exists t_1 \geqslant t_0$ such that $q_t(v) \leqslant R'$. We choose an infinitely bounded set W, maximal for inclusion, that contains a node in $\mathcal{S} \cup \mathcal{D}$. The size of W is defined as the sum of the queue sizes of the nodes it contains.

Since all nodes in W are infinitely bounded, it exists an infinite number of times $\{t_i\}_{i \in \mathbb{N}}$ such that the size of W is minimum between t_i and t_{i+1}, and with $q_{t_i}(w) \leqslant R'$ for all $w \in W$. The number of stored packets in W at time $t_i - 1$ is thus strictly greater than the one at time t_i: $\sum_{w \in W} q_{t_i}(w) < \sum_{w \in W} q_{t_i - 1}(w)$.

The growth of the size of W between steps $t_i - 1$ and t_i is the following:

$$\begin{aligned} \sum_{w \in W} q_{t_i - 1}(w) = & \sum_{w \in W} q_{t_i}(w) + \sum_{w \in W} out_{t_i - 1}(w) - \sum_{w \in W} in_{t_i - 1}(w) \\ & -|\{(u, v) \in E_{t_i - 1}, u \notin W, v \in W\}| \\ & +|\{(u, v) \in E_{t_i - 1}, u \in W, v \notin W\}|, \end{aligned}$$

where $E_{t_i - 1}$ is the set of links used by LGG at time $t_i - 1$.

Moreover, $\sum_{w \in W} out_{t_i - 1}(w) \leqslant \sum_{w \in W} out(w)$ from the definition af the R-generalized destinations, and $\sum_{w \in W} in_{t_i - 1}(w) \leqslant \sum_{w \in W} in(w)$ from the assumption made at the beginning of this section.

We thus get:

$$\begin{aligned}\sum_{w\in W} q_{t_i}(w) < \; & \sum_{w\in W} q_{t_i-1}(w) = \sum_{w\in W} q_{t_i}(w) + \sum_{w\in W} out(w) - \sum_{w\in W} in(w) \\ & -|\{(u,v)\in E_{t_i-1}, u\notin W, v\in W\}| \\ & +|\{(u,v)\in E_{t_i-1}, u\in W, v\notin W\}|\end{aligned}$$

which is equivalent to:

$$\begin{aligned}\sum_{w\in W} out(w) > \; & \sum_{w\in W} in(w) \\ & +|\{(u,v)\in E_{t_i-1}, u\notin W, v\in W\}| \\ & -|\{(u,v)\in E_{t_i-1}, u\in W, v\notin W\}|\end{aligned}$$

Since cut $((V\cup\{s^*\}),\{d^*\})$ is minimum, $\sum_{w\in W} out(w) < \sum_{w\in W} in(w) + |C|$, where C is the set of links incident to W in G. So $|C| = |\{(u,v)\in E_{t_i-1}, u\notin W, v\in W\}| + |\{(u,v)\in E_{t_i-1}, u\in W, v\notin W\}|$, which leads to:

$$\begin{aligned}\sum_{w\in W} out(w) < \; & \sum_{w\in W} in(w) \\ & +|\{(u,v)\in E_{t_i-1}, u\notin W, v\in W\}| \\ & +|\{(u,v)\in E_{t_i-1}, u\in W, v\notin W\}|\end{aligned}$$

Thus, the number of packets sent at time $t_i - 1$ from a node in W to a node in $V\setminus W$ is strictly positive. This means that it exists at least a node $w\in V\setminus W$ whose height is lower that the one of a node in W. We thus get a contradiction since W was assumed maximum for inclusion.

Whatever W chosen, we find a node in nœud de $V\setminus W$ infinitely bounded. As W is infinitely bounded and the network size is finite, it exists a node v^* infinitely bounded such that $W\cup\{v^*\}$ is infinitely bounded. In that way, we show that V is infinitely bounded.

Thus, all nodes in G have a queue of bounded size ($q_t(v)\leqslant R'$) an infinite number of times. Since the number of injected packets at each time step t is equal to the capacity of extraction of the R-generalized sources/destinations, then the number of stored packets in the network never decreases. We can therefore conclude that the number of stored packets stays bounded for all t.

This concludes the proof of the stability of LGG when the R-generalized $\mathcal{S}$-$\mathcal{D}$-network G is saturated at the destinations, according to the correctness of Conjecture 1.

3.3 Saturated R-generalized $\mathcal{S}$-$\mathcal{D}$-network

Here, (A,B) is a minimum cut of value $\sum_{v\in\mathcal{S}\cup\mathcal{D}} in(v)$, such that $|A|,\ |B|>1$ and $|A|,\ |B|<|V|$. We successively prove that partitions A and B can be viewed as two different generalized networks. Our induction hypothesis is the following:

Our protocol LGG is stable on all R'-generalized $\mathcal{S}'$-$\mathcal{D}'$-network of n nodes, $\forall R' \geqslant 0$, $n < |V|$, with $|V| > 1$.

In the rest of the section, we show by induction, using this hypothesis, that LGG is stable on G.

3.3.1 The number of packets stored in B is bounded

We show that partition B of the cut (A, B) is a feasible R-generalized $\mathcal{S}'$-$\mathcal{D}'$-network, and then, that the number of packets stored in B is bounded.

We construct a $\mathcal{S}'$-$\mathcal{D}'$-network B' that acts as B in G. Let X be the set of nodes in B adjacent to a node in partition A. Consider the network B' containing a set $\mathcal{S}'$ of R-generalized sources, and a set $\mathcal{D}'$ of R-generalized destinations, such that $\mathcal{S}' \cup \mathcal{D}' = X \cup (\mathcal{D} \cap B) \cup (\mathcal{S} \cap B)$, defined in the following way:

- each node in $B \setminus X$ keeps the same behaviour in B' as the one in B (and thus in G);
- each node $v \in X \setminus (\mathcal{S} \cup \mathcal{D})$ becomes a R-generalized source of $\mathcal{S}'$ with $in_{B'}(v) = |\Gamma_{|A}(v)|$, and $out_{B'}(v) = 0$;
- parameters $in(v)$ and $out(v)$ of nodes in $X \cap (\mathcal{S} \cup \mathcal{D})$ are updated in B' and respectively become $in_{B'}(v) = in(v) + |\Gamma_{|A}(v)|$, and $out_{B'}(v) = out(v)$. If $in_{B'}(v) \geqslant out_{B'}(v)$, then $v \in \mathcal{S}'$, sinon $v \in \mathcal{D}'$.

By induction hypothesis, if the R-generalized $\mathcal{S}'$-$\mathcal{D}'$-network B' is feasible, then LGG is stable on B'. Indeed, by definition of B', we can choose a number of injected packets into nodes of X in the way that B' acts as B in G. If LGG is stable on B', then it is stable on B.

Since (A, B) has value $\sum_{v \in \mathcal{S} \cup \mathcal{D}} in(v)$, each link on the border of the cut transmit one unit of flow Φ. By definition of a R-generalized source and by well choosing the caracteristics of the nodes of X, the flow $\Phi_{B'}$ that injects $in(v)$ packets in each R-generalized source/destination $v \in \mathcal{S} \cup \mathcal{D}$, and that follows links used by Φ in G is feasible. Indeed, from the Kirchhoff laws respected by the flow, $\Phi_{B'}$ is lower than or equal to Φ n each link used by Φ.

We have found a feasible flow $\Phi_{B'}$ in B', thus the R-generalized $\mathcal{S}'$-$\mathcal{D}'$-network B' is feasible, and then the number of stored packets in B is bounded. Let R_B be the maximum number of packets stored in B.

3.3.2 The number of packets stored in A is bounded

We use the same reflection to show that the number of stored packets in partition A is bounded. A can be viewed as a R_B-generalized $\mathcal{S}''$-$\mathcal{D}''$-network A' in which $\mathcal{S}'' \cup \mathcal{D}'' = Y \cup (\mathcal{D} \cap A) \cup (\mathcal{S} \cap A)$, with Y the set of nodes in A adjacent to some node in B. We then prove that A' is feasible in order to bound the number of stored packets in it.

Sets $\mathcal{S}''$ and $\mathcal{D}''$ of R_B-generalized sources and destinations are defined in the following way:

- each node in $A \setminus Y$ keeps the same behaviour A' as in A (and then in G);
- each node $v \in Y \setminus (\mathcal{S} \cup \mathcal{D})$ becomes a R_B-generalized destination of $\mathcal{D}''$ with $out_{A'}(v) = |\Gamma_{|B}(v)|$, and $in_{A'}(v) = 0$;
- finally, the parameters $in(v)$ and $out(v)$ of nodes $v \in Y \cap (\mathcal{S} \cup \mathcal{D})$ are updated in A' and become respectively $in_{A'}(v) = in(v)$, and $out_{B'}(v) = out(v) + |\Gamma_{|B}(v)|$. If $in_{A'}(v) \geqslant out_{A'}(v)$, then $v \in \mathcal{S}''$, otherwise $v \in \mathcal{D}''$.

Remark 2 *Remark that $\mathcal{D}'' \neq \emptyset$. Indeed, if it not the case, then $\sum_{v \in (\mathcal{S} \cup \mathcal{D})} in(v) > \sum_{v \in Y} |\Gamma_{|B}(v)| = \sum_{v \in (\mathcal{S} \cup \mathcal{D})} \Phi(s^*, v)$ and we obtain a contradiction according to the existence of a feasible flow Φ in G.*

This remark allows us to apply the induction hypothesis and conclude that if the R_B-generalized $\mathcal{S}''$-$\mathcal{D}''$-network A' is feasible, then LGG is stable on A'. Then, as LGG is stable on A' and Y has the same behaviour in A' and in A, the stability of LGG on A is complete. As for B, flow Φ restricted to the nodes in A' is a feasible flow in A' extended. Therefore, the number of paquets stored in A stays bounded.

This concludes the general proof of Theorem 2 on the stability of LGG.

4 Conclusion

In this paper, we show that our protocol LGG is stable in a $\mathcal{S}$-$\mathcal{D}$-network in which the number of injected packets at each time step is lower than or equal to the value of a maximum flow in the network, according to the correctness of Conjecture 1 in the special case where the flow is constrained at the destination nodes. This work allows to consider several perspectives. We present the main conjectures that follow from our study and point to interesting results on the stability of queueing systems.

Considering the initial case of a classical $\mathcal{S}$-$\mathcal{D}$-network. If the arrival rate changes at each time step, then we assume the following result:

Conjecture 2 *If the arrival rate generated at each time step t exceeds the available capacity, i.e. the value of a maximum flow, then it must be some time after t to extracts the stored excess.*

This condition is necessary and sufficient to ensure the stability of LGG on the network.

More formally, let G be a $\mathcal{S}$-$\mathcal{D}$-network, Φ a maximum s^*-d^*-flow in G^* with value f^*, and $in_t(s)$ the number of injected packets in queue of source $s \in \mathcal{S}$ at time t. The stability condition would thus be:

For all t and dt, if $\sum_{s \in \mathcal{S}} \sum_{k=1}^{dt} in_{t+k}(s) > dt \cdot f^*$, then it exists a time t' such that $\sum_{s \in \mathcal{S}} \sum_{k=1}^{t'} in_{t+k}(s) \leqslant (t'-t) \cdot f^*$.

The idea of the proof would thus be to consider that all the generated packets when the number of already injected packets is important, will actually be generated later.

If we now consider the case where the arrival rate in the network follows a uniform distribution, then we conjecture the following result:

Conjecture 3 *If the number of injected packets $in_t(s)$ at time t in the queue of source $s \in \mathcal{S}$ follows a uniform distribution of mean strictly less than than the value of a minimum $\mathcal{S}$-$\mathcal{D}$-cut, then with high probability our protocol is stable on the $\mathcal{S}$-$\mathcal{D}$-network.*

The case of a dynamic network in which the topology (nodes and links) changes among time, is an important perspective of research. The stability of LGG in these networks might depend on the existence of a feasible flow in the network. In other words:

Conjecture 4 *If the number of injected packets ensures the existence of a feasible $\mathcal{S}$-$\mathcal{D}$-flow, then our protocol LGG is stable on the network, at least in the unsaturated case.*

Finally, an assumptions made in this work is that there is no interference among simultaneous transmissions in the network. In order to deal with wireless interferences, we have to compute, for each step of our algorithm LGG, the set of pairwise compatible links E_t. The goal is to find, at each time step t, the optimal set E_t in order to guaranty that the number of stored packets in the $\mathcal{S}$-$\mathcal{D}$-network stays bounded:

Conjecture 5 *If an oracle can provide an optimal set E_t in the $\mathcal{S}$-$\mathcal{D}$-network G at time t, then LGG is stable on G.*

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